

TÜREV ALMA KURALLARI BÖLÜM 5

LOGARİTMA FONKSİYONUNUN TÜREVİ HATIRLATMA

$a \in \mathbb{R}^+ - \{1\}$ olmak üzere,
 $f: \mathbb{R}^+ \rightarrow \mathbb{R}, y = f(x) = \log_a x$ fonksiyonuna
logaritma fonksiyonu demiştik. Özel
olarak $a = 2,71... = e$ irrasyonel sayısı ise
 $\log_e x = \ln x$ olarak gösteriyorduk.

$$e = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = 2,71... \text{ dir.}$$

TEOREM

$$f: \mathbb{R}^+ \rightarrow \mathbb{R}, y = f(x) = \ln x \text{ ise } y' = f'(x) = \frac{1}{x} \text{ tir.}$$

GENELLEME

$$f: A \rightarrow \mathbb{R}, y = \ln u \text{ ise } y' = \frac{u'}{u} \text{ dur.}$$

Örnek...1 :

$$f(x) = \ln(x^2 - 9x) \text{ ise } f'(x) = ?$$

$$f'(x) = \frac{(x^2 - 9x)'}{x^2 - 9x} = \frac{2x - 9}{x^2 - 9x} //$$

Örnek...2 :

$$f(x) = x \cdot \ln(x) \text{ ise } f'(x) = ?$$

$$f'(x) = (x)' \cdot \ln x + x \cdot (\ln x)'$$

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$f'(x) = \ln x + 1 //$$

Örnek...3 :

$$f(x) = \ln(\sin x) \text{ ise } f'(x) = ?$$

$$f'(x) = \frac{(\sin x)'}{\sin x} = \frac{\cos x}{\sin x} = \cot x //$$

Örnek...4 :

$$f(x) = \log(\sqrt[3]{x^3}) \text{ ise } f'(x) = ?$$

$$f(x) = \log(x)^{\frac{3}{3}} \Rightarrow f(x) = \frac{3}{3} \cdot \log x \Rightarrow f(x) = \frac{3}{3} \cdot \log_{10} x$$

Taban değiştirerek doğal logaritma yapalım.

$$f(x) = \frac{3}{3} \cdot \frac{\log_e x}{\log_{10} e} \Rightarrow f(x) = \frac{3}{7 \cdot \ln 10} \cdot \ln x$$

$$f'(x) = \frac{3}{7 \cdot \ln 10} \cdot \frac{1}{x} //$$

Örnek...5 :

$$f(x) = \ln(\arctan x) \text{ ise } f'(1) = ?$$

$$f'(x) = \frac{(\arctan x)'}{\arctan x} \Rightarrow f'(x) = \frac{1}{1+x^2} \cdot \frac{1}{\arctan x}$$

$$f'(1) = \frac{1}{(1+1^2) \cdot \arctan 1} //$$

Örnek...6 :

$$f(x) = \log_2(\cos x) \text{ ise } f'(x) = ?$$

NOT

$$y = \log_a u(x) \text{ ise}$$

$$y' = \frac{1}{\ln a} \cdot \frac{u'(x)}{u(x)} \text{ olur.}$$

$$f(x) = \log_2(\cos x) \text{ ise}$$

$$f'(x) = \frac{1}{\ln 2} \cdot \frac{(\cos x)'}{\cos x}$$

$$f'(x) = \frac{-\sin x}{\ln 2 \cdot \cos x}$$

$$f'(x) = \frac{-\tan x}{\ln 2}$$

Örnek...7 :

$$f(x) = \log_3\left(\frac{x}{\tan x}\right) \text{ ise } f'(x) = ?$$

Önce logaritma özelliğinden yararlanarak bölmeyi çıkarmaya. çevirip soruyu kolaylaştıralım.

$$f(x) = \log_3 x - \log_3 \tan x$$

$$f'(x) = \frac{1}{\ln 3} \cdot \frac{1}{x} - \frac{1}{\ln 3} \cdot (1 + \tan^2 x) \Rightarrow f'(x) = \frac{1}{x \cdot \ln 3} - \frac{1 + \tan^2 x}{\ln 3} //$$

Örnek...8 :

$$f(x) = \log(4x^3 - 25) \text{ ise } f'(x) = ?$$

$$f'(x) = \frac{1}{\ln 10} \cdot \frac{12x^2}{4x^3 - 25} //$$

TÜREV - 5

(LOGARİTMA VE ÜSTEL FONKSİYON TÜREV ALMA KURALLARI)

DEĞERLENDİRME

1) $f(x)=\ln(x^2-4x+5)$ ise $f'(x)=?$

$$f'(x) = \frac{2x-4}{x^2-4x+5} //$$

2) $f(x)=\ln^3(x^2)$ ise $f'(e)=?$

$$\begin{aligned} f(x) &= [\ln(x^2)]^3 \\ f'(x) &= 3 \cdot [\ln(x^2)]^2 \cdot (\ln(x^2))' \\ f'(x) &= 3 \cdot [\ln(x^2)]^2 \cdot \frac{2x}{x^2} \\ f'(e) &= 3 \cdot [\ln(e^2)]^2 \cdot \frac{2e}{e^2} \\ f'(e) &= 3 \cdot [2]^2 \cdot \frac{2}{e} = \frac{24}{e} // \end{aligned}$$

3) $f(x)=\ln(x^2 \cdot \cos(x-1))$ ise $f'(1)=?$

$$\begin{aligned} f'(x) &= \frac{(x^2 \cdot \cos(x-1))'}{x^2 \cdot \cos(x-1)} \\ f'(x) &= \frac{2x \cdot \cos(x-1) + x^2 \cdot [-\sin(x-1)]}{x^2 \cdot \cos(x-1)} \\ f'(1) &= \frac{2 \cdot \cos(0) + 1 \cdot [-\sin(0)]}{1 \cdot \cos(0)} = \frac{2 \cdot 1 + 1 \cdot 0}{1 \cdot 1} \\ f'(1) &= \frac{2}{1} = 2 // \end{aligned}$$

4) $f(x)=\ln|\log x|$ ise $f'(10)=?$

$$\begin{aligned} f'(x) &= \frac{(\log x)'}{\log x} = \frac{\frac{1}{\ln 10} \cdot \frac{1}{x}}{\log x} \\ f'(x) &= \frac{1}{\ln 10 \cdot x \cdot \log x} // \end{aligned}$$

5) $f(x)=\log_2\left(\frac{x^2+3x}{2x-5}\right)$ ise $f'(3)=?$

$$\begin{aligned} f'(x) &= \frac{1}{\ln 2} \cdot \frac{\left(\frac{x^2+3x}{2x-5}\right)'}{\left(\frac{x^2+3x}{2x-5}\right)} = \frac{1}{\ln 2} \cdot \frac{(2x+3) \cdot (2x-5) - (x^2+3x) \cdot (2)}{(2x-5)^2} \\ f'(3) &= \frac{1}{\ln 2} \cdot \frac{8 \cdot 1 - 18 \cdot 2}{1^2} \\ f'(3) &= \frac{1}{\ln 2} \cdot \frac{-27}{1} \Rightarrow f'(3) = \frac{-27}{\ln 2} // \end{aligned}$$

6) $f(x)=\log\left(\frac{x}{e}\right)$ ise $f'(1)=?$

$$\begin{aligned} f(x) &= \log x - \log e = \log_{10} x - \ln e \\ f'(x) &= \frac{1}{\ln 10} \cdot \frac{1}{x} - 0 \\ f'(1) &= \frac{1}{\ln 10} \cdot \frac{1}{1} \\ f'(1) &= \frac{1}{\ln 10} // \end{aligned}$$

7) $f(x+\arctan 2x)=\ln(x^2+4x+1)$ ise $(f^{-1})'(0)=?$ ($0 \leq x$)

f karsıya f^{-1} olarak geçer.

$$x + \arctan 2x = f^{-1}(\ln(x^2+4x+1))$$

Her iki tarafın türevini alalım.

$$1 + \frac{2}{1+4x^2} = [f^{-1}(\ln(x^2+4x+1))]' \cdot [\ln(x^2+4x+1)]'$$

$$1 + \frac{2}{1+4x^2} = [f^{-1}(\ln(x^2+4x+1))]' \cdot \frac{2x+4}{x^2+4x+1}$$

$$[f^{-1}(\ln(x^2+4x+1))]' = \left(1 + \frac{2}{1+4x^2}\right) \cdot \frac{x^2+4x+1}{2x+4} \Rightarrow$$

$$\begin{aligned} \ln(x^2+4x+1) &= 0 \\ x^2+4x+1 &= e^0 \\ x^2+4x+1 &= 1 \\ x^2+4x &= 0 \\ x(x+4) &= 0 \\ x=0 \quad x=-4 \\ x \geq 0 \text{ dediğ'i} \\ \text{İçin } x=0 \\ \text{alınır.} \end{aligned}$$

8) $f(x)=\log(\sqrt[3]{\sin x})$ ise $f'(\frac{\pi}{4})=?$

$$\begin{aligned} f(x) &= \log(\sin x)^{\frac{1}{3}} \\ f'(x) &= \frac{1}{3} \cdot \log(\sin x) \\ f'(x) &= \frac{1}{3} \cdot \frac{1}{\ln 10} \cdot \frac{\cos x}{\sin x} = \frac{1}{3} \cdot \frac{1}{\ln 10} \cdot \cot x \\ f'(\frac{\pi}{4}) &= \frac{1}{3} \cdot \frac{1}{\ln 10} \cdot \cot(\frac{\pi}{4}) \\ f'(\frac{\pi}{4}) &= \frac{1}{3} \cdot \frac{1}{\ln 10} \cdot 1 = \frac{1}{3 \cdot \ln 10} // \end{aligned}$$

TÜREV - 5

(LOGARİTMA VE ÜSTEL FONKSİYON TÜREV ALMA KURALLARI)

ÜSTEL FONKSİYONUNUN TÜREVİ

$a \in \mathbb{R}^+ - \{1\}$ olmak üzere,
 $f: \mathbb{R} \rightarrow \mathbb{R}^+, y=f(x)=a^x$ fonksiyonuna üstel
fonksiyon demiştik.

TEOREM

$f: \mathbb{R} \rightarrow \mathbb{R}^+, y=f(x)=a^x$ ise $y'=a^x \cdot \ln a$ dır.

GENELLEME

$y=a^u$ ise $y'=a^u \cdot u' \cdot \ln a$ dır.

Örnek...1 :

$y=f(x)=5^x$ ise $f'(x)=?$ $5^x \cdot \ln 5 //$

Örnek...2 :

$y=f(x)=2^{3x+4}$ ise $f'(0)=?$

$$f'(x) = (3x+4)' \cdot 2^{3x+4} \cdot \ln 2$$

$$f'(x) = 3 \cdot 2^{3x+4} \cdot \ln 2$$

$$f'(0) = 3 \cdot 2^{0+4} \cdot \ln 2 = 3 \cdot 16 \cdot \ln 2$$

$$f'(0) = 48 \cdot \ln 2 //$$

Örnek...3 :

$y=f(x)=7^{\tan x} + \cos^2 x$ ise $f'(\frac{\pi}{4})=?$

$$f'(x) = (\tan x)' \cdot 7^{\tan x} \cdot \ln 7 + 2 \cdot (\cos x) \cdot (-\sin x)$$

$$f'(x) = (1 + \tan^2 x) \cdot 7^{\tan x} \cdot \ln 7 - \sin 2x$$

$$f'(\frac{\pi}{4}) = (1 + \tan^2 \frac{\pi}{4}) \cdot 7^{\tan \frac{\pi}{4}} \cdot \ln 7 - \sin(2 \cdot \frac{\pi}{4})$$

$$= (1 + 1^2) \cdot 7^1 \cdot \ln 7 - \sin(\frac{\pi}{2})$$

$$= 2 \cdot 7 \cdot \ln 7 - 1$$

$$= 14 \cdot \ln 7 - 1 //$$

Örnek...4 :

$y=f(x)=\left(\frac{1}{9}\right)^{x^3+x-1}$ ise $f'(x)=?$

$$f'(x) = (3x^2+2x) \cdot \left(\frac{1}{9}\right)^{x^3+x-1} \cdot \ln\left(\frac{1}{9}\right) //$$

Örnek...5 :

$y=f(x)=e^x$ ise $f'(x)=?$

$$f'(x) = e^x \cdot \ln e$$

$$f'(x) = e^x //$$

Örnek...6 :

k sabit bir sayı olmak üzere, $y=f(x)=e^{kx}$ ise $f'(x)=?$

$$f'(x) = k \cdot e^{kx} \cdot \ln e$$

$$f'(x) = k \cdot e^{kx} //$$

Örnek...7 :

$y=f(x)=e^{x^3-1}$ ise $f'(1)=?$

$$f'(x) = 3x^2 \cdot e^{x^3-1} \cdot \ln e = 1$$

$$f'(1) = 3 \cdot 1^2 \cdot e^{1^3-1}$$

$$f'(1) = 3 \cdot e^0 = 3 //$$

Örnek...8 :

$y=f(x)=e^{x^2-\sin^2 x}$ ise $f'(0)=?$

$$f'(x) = (x^2 - \sin^2 x)' \cdot e^{x^2-\sin^2 x} \cdot \ln e = 1$$

$$f'(x) = (2x - 2 \sin x \cdot \cos x) \cdot e^{x^2-\sin^2 x}$$

$$f'(0) = (0 - 2 \sin 0 \cdot \cos 0) \cdot e^{0^2-\sin^2 0}$$

$$f'(0) = 0 \cdot e^0 = 0 \cdot 1 = 0 //$$

Y=[U(X)]^{V(X)} BİÇİMİNDEKİ İFADELER

Bu gibi ifadelerin türevi iki adımda alınır.

Adım 1) Her iki tarafı logaritması alınır.

Adım 2) Çarpım Kuralına göre türev alınır.

Örnek...9 :

$$y=f(x)=(\cos x)^{\sin x} \text{ ise } f'(x)=?$$

$$y = (\cos x)^{\sin x} \text{ (her iki tarafın ln'ini alalım.)}$$

$$\ln y = \ln (\cos x)^{\sin x}$$

$$\ln y = \sin x \cdot \ln (\cos x) \text{ (her iki tarafın türevini alalım)}$$

$$\frac{y'}{y} = \cos x \cdot \ln (\cos x) + \sin x \cdot \left(\frac{-\sin x}{\cos x} \right) \text{ (işler dışlar ile y'ns yalnız bırakalım)}$$

$$y' = y \cdot \left[\cos x \cdot \ln (\cos x) - \frac{\sin^2 x}{\cos x} \right]$$

$$y' = (\cos x)^{\sin x} \cdot \left[\cos x \cdot \ln (\cos x) - \frac{\sin^2 x}{\cos x} \right] \text{ (y yerine ilk boşluğu değeri yazdık)}$$

Örnek...10 :

$$y=f(x)=(x^3+x^2)^{x+1} \text{ ise } f'(x)=?$$

$$\ln y = \ln (x^3+x^2)^{x+1}$$

$$\ln y = (x+1) \cdot \ln (x^3+x^2)$$

$$\frac{y'}{y} = 1 \cdot \ln (x^3+x^2) + (x+1) \cdot \left(\frac{3x^2+2x}{x^3+x^2} \right)$$

$$y' = (x^3+x^2)^{x+1} \cdot \left[1 \cdot \ln (x^3+x^2) + (x+1) \cdot \frac{3x^2+2x}{x^3+x^2} \right]$$

Örnek...11 :

$$y=f(x)=(x)^x \text{ ise } f'(x)=?$$

$$y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \cdot \ln x$$

$$\frac{y'}{y} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$

$$y' = y \cdot (\ln x + 1)$$

$$y' = x^x \cdot (\ln x + 1) //$$

Örnek...12 :

$$y=f(x)=(\ln x)^x \text{ ise } f'(x)=?$$

$$y = (\ln x)^x$$

$$\ln y = \ln (\ln x)^x$$

$$\ln y = x \cdot \ln (\ln x)$$

$$\frac{y'}{y} = 1 \cdot \ln (\ln x) + x \cdot \left[\ln (\ln x) \right]'$$

$$\frac{y'}{y} = \ln (\ln x) + x \cdot \frac{(\ln x)'}{\ln x}$$

$$\frac{y'}{y} = \ln (\ln x) + x \cdot \frac{\frac{1}{x}}{\ln x}$$

Örnek...13 :

$$y=f(x)=(\arctan x)^x \text{ ise } f'(1)=?$$

$$y = (\arctan x)^x$$

$$\ln y = \ln (\arctan x)^x$$

$$\ln y = x \cdot \ln (\arctan x)$$

$$\frac{y'}{y} = 1 \cdot \ln (\arctan x) + x \cdot \frac{\frac{1}{1+x^2}}{\arctan x}$$

$$y' = y \cdot \left(\ln (\arctan x) + \frac{x}{(1+x^2) \cdot \arctan x} \right)$$

$$y' = (\arctan x)^x \cdot \left(\ln (\arctan x) + \frac{x}{(1+x^2) \cdot \arctan x} \right) //$$

DEĞERLENDİRME

1) $y=f(x)=2^{\ln^2(x+1)}$ ise $f'(e-1)=?$

$$f'(x) = (\ln^2(x+1))' \cdot 2^{\ln^2(x+1)} \cdot \ln 2$$

$$f'(x) = \left(2 \cdot \ln(x+1) \cdot \frac{1}{x+1} \right) \cdot 2^{\ln^2(x+1)} \cdot \ln 2$$

$$f'(e-1) = \left(2 \cdot \ln(e-x+1) \cdot \frac{1}{e-x+1} \right) \cdot 2^{\ln^2(e-x+1)} \cdot \ln 2$$

$$f'(e-1) = \left(2 \cdot 1 \cdot \frac{1}{e} \right) \cdot 2^1 \cdot \ln 2 = \frac{4 \cdot \ln 2}{e} //$$

2) $y=f(x)=\frac{2^{x^2+4}}{\ln 4}$ ise $f'(0)=?$

$$f'(x) = \frac{1}{\ln 4} \left[(2^x) \cdot 2^{x^2+4} \cdot \ln 2 \right]$$

$$f'(0) = \frac{1}{\ln 4} \left[(2^0) \cdot 2^{0+4} \cdot \ln 2 \right] = \frac{0}{\ln 4} = 0 //$$

3) $y=f(x)=e^x \cdot \ln x$ ise $f'(1)=?$

$$f'(x) = (e^x)' \cdot \ln x + e^x \cdot (\ln x)'$$

$$f'(x) = e^x \cdot \ln x + e^x \cdot \frac{1}{x}$$

$$f'(1) = e^1 \cdot \ln 1 + e^1 \cdot \frac{1}{1}$$

$$f'(1) = e \cdot 0 + e = e //$$

4) $y=f(x)=2^{g(x^2)}$, $g'(4)=\frac{1}{512}$ ve $g(4)=8$ ise

$$f'(2)=?$$

$$f'(x) = [g(x^2)]' \cdot 2^{g(x^2)} \cdot \ln 2 = g'(x^2) \cdot (2x) \cdot 2^{g(x^2)} \cdot \ln 2$$

$$f'(2) = g'(4) \cdot (4) \cdot 2^{g(4)} \cdot \ln 2$$

$$f'(2) = \frac{1}{512} \cdot 4 \cdot 2^8 \cdot \ln 2 = \frac{1}{28} \cdot 2^8 \cdot \ln 2$$

$$f'(2) = 2 \cdot \ln 2 //$$

5) $y=f(x)=e^x \cdot \arctan x$ ve $g(x)=\arcsin x$ ise $(g \circ f)'(0)=?$

$$(g \circ f)'(x) = [g(f(x))]' = g'(f(x)) \cdot f'(x)$$

$$(g \circ f)'(x) = \frac{1}{\sqrt{1-(e^x \arctan x)^2}} \cdot \left(e^x \arctan x + \frac{e^x}{1+x^2} \right)$$

$$f'(x) = e^x \arctan x + e^x \cdot \frac{1}{1+x^2} \quad (g \circ f)'(0) = \frac{1}{\sqrt{1-(1 \cdot \arctan 0)^2}} \cdot \left(1 \cdot \arctan 0 + \frac{1}{1+0} \right)$$

$$g(x) = \arcsin x$$

$$g'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$g'(f(x)) = \frac{1}{\sqrt{1-(e^x \arctan x)^2}}$$

$$(g \circ f)'(0) = \frac{1}{\sqrt{1-0^2}} \cdot (1 \cdot 0 + 1)$$

$$(g \circ f)'(0) = \frac{1}{1} \cdot 1 = 1 //$$

6) $y=f(x)=\left(\frac{1}{x}\right)^x$ ise $f'(x)=?$

$$y = (x^{-1})^x$$

$$y = x^{-x}$$

$$\ln y = \ln(x)^{-x}$$

$$\ln y = (-x) \cdot \ln x$$

$$\frac{y'}{y} = (-1) \cdot \ln x + (-x) \cdot \frac{1}{x}$$

$$\frac{y'}{y} \neq -\ln x - 1 \Rightarrow y' = y \cdot (-\ln x - 1)$$

$$\frac{y'}{y} \neq -\ln x - 1 \Rightarrow y' = \left(\frac{1}{x}\right)^x \cdot (-\ln x - 1) //$$

7) $y=f(x)=(x)^{\ln x}$ ise $f'(x)=?$

$$y = x^{\ln x}$$

$$\ln y = \ln(x)^{\ln x}$$

$$\ln y = \ln x \cdot \ln x$$

$$\ln y = (\ln x)^2$$

$$\frac{y'}{y} = 2 \cdot \ln x \cdot \frac{1}{x}$$

$$y' = y \cdot 2 \cdot \ln x \cdot \frac{1}{x} \Rightarrow y' = (x)^{\ln x} \cdot 2 \cdot \ln x \cdot \frac{1}{x} //$$

8) $y=f(x)=(\tan x)^{\cot x}$ ise $f'\left(\frac{\pi}{4}\right)=?$

$$y = \tan x^{\cot x}$$

$$\ln y = \ln(\tan x)^{\cot x}$$

$$\ln y = \cot x \cdot \ln(\tan x)$$

$$\frac{y'}{y} \neq (-1 - \cot^2 x) \cdot \ln(\tan x) + \cot x \cdot \left(\frac{1 + \tan^2 x}{\tan x} \right)$$

$$y' = (\tan x)^{\cot x} \cdot \left[(-1 - \cot^2 x) \cdot \ln(\tan x) + \cot x \cdot \left(\frac{1 + \tan^2 x}{\tan x} \right) \right]$$

$$f'\left(\frac{\pi}{4}\right) = (1)^1 \cdot \left[(-1 - 1^2) \cdot \ln(1) + \frac{1 \cdot (1 + 1^2)}{1} \right]$$

$$f'\left(\frac{\pi}{4}\right) = 1 \cdot [(-2) \cdot 0 + 2] = 2 //$$

9) $y=f(x)=(1+x)^{x+1}$ ise $f'(0)=?$

$$y = (x+1)^{x+1}$$

$$\ln y = \ln(x+1)^{x+1}$$

$$\ln y = (x+1) \cdot \ln(x+1)$$

$$\frac{y'}{y} = 1 \cdot \ln(x+1) + (x+1) \cdot \frac{1}{x+1}$$

$$y' = (x+1)^{x+1} \cdot [\ln(x+1) + 1]$$

$$f'(0) = 1^1 \cdot [\ln(1) + 1]$$

$$f'(0) = 1 //$$