

İNTEGRAL-4

BELİRLİ İNTEGRAL

BELİRLİ İNTEGRAL KURALLARI

$$1. \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$2. \int_a^a f(x) dx = 0$$

$$3. \int_a^b k \cdot f(x) dx = k \cdot \int_a^b f(x) dx$$

$$4. \int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

$$5. \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx, \quad c < b$$

$$6. (b-a) \cdot \min(f(x)) < \int_a^b f(x) dx = 0 < (b-a) \cdot \max(f(x))$$

$$7. f: [a, b] \rightarrow \mathbb{R} \text{ sürekli bir fonksiyon ve } \frac{d}{dx}(F(x)) = f(x) \text{ ise } \int_a^b f(x) dx = F(b) - F(a) \text{ olur.}$$

Örnek...1 :

$$\int_2^3 f(x) dx = 4 \text{ ve } \int_3^7 f(x) dx = 9 \text{ ise } \int_2^7 f(x) dx = ?$$

$$\begin{array}{r} - / F(3) - F(2) = 4 \\ + - / F(7) - F(3) = 9 \\ \hline F(7) - F(2) = 13 \end{array}$$

Örnek...2 :

$$\text{Uygun koşullarda } \int_2^5 f(x) dx = 4 \text{ ve } \int_4^7 f(x) dx = 9 \text{ ve}$$

$$\int_2^7 f(x) = 21 \text{ ise } \int_4^5 f(x) = ?$$

$$\begin{array}{r} F(5) - F(2) = 4 \\ F(7) - F(4) = 9 \\ \hline F(7) - F(2) = 21 \end{array} \quad \begin{array}{l} F(5) - F(4) = ? \\ \hline F(5) - F(4) = 13 - 21 = -8 \end{array}$$

Örnek...3 :

$$\int_1^2 x dx = ? \quad \frac{x^2}{2} \Big|_1^2 = \frac{2^2 - 1^2}{2} = \frac{3}{2}$$

Örnek...4 :

$$\int_{-1}^0 (x^3 + 5x^2 - 7x + 4) dx = ?$$
$$\frac{x^4}{4} + \frac{5x^3}{3} - \frac{7x^2}{2} + 4x \Big|_{-1}^0$$
$$0 - \left(\frac{1}{4} - \frac{5}{3} - \frac{7}{2} - 4 \right)$$
$$= \frac{107}{12}$$

Örnek...5 :

$$\int_0^2 \left(3x^2 + \frac{x}{3} \right) dx = ?$$
$$\frac{3x^3}{3} + \frac{x^2}{6} \Big|_0^2$$
$$2 + \frac{4}{6} = \frac{26}{3}$$

UYARI

Değişken değiştirme yapıldığında yeni değişkene göre sınırlar tekrar hesaplanırsa eski değişkene dönülmeden integral hesaplanabilir

Örnek...6 :

$$\int_2^3 (x-2)^2 dx = ?$$
$$u = x-2 \quad du = dx$$
$$\int_0^1 u^2 du = \frac{u^3}{3} \Big|_0^1 = \frac{1}{3}$$

Örnek...7 :

$$\int_0^1 (x^2 + 5x + 1)^2 (2x + 5) dx = ?$$
$$u = x^2 + 5x + 1 \quad du = 2x + 5$$
$$\int_1^7 u^2 du = \frac{u^3}{3} \Big|_1^7 = \frac{7^3 - 1^3}{3} = \frac{342}{3}$$

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Örnek...8 :

$$\int_{-2}^2 x^5 + x^3 dx = ? \quad \frac{x^6}{6} + \frac{x^4}{4} \Big|_{-2}^2 = 0$$

* $\int_{-a}^a \text{tek fonksiyon} = 0$

Örnek...9 :

$$\int_{-3}^3 \frac{x^{15}}{1+x^4} dx = ? = 0$$

tek fonksiyon

Not $f(x) = \frac{x^{15}}{1+x^4}$
 $f(-x) = \frac{(-x)^{15}}{1+(-x)^4} = \frac{-x^{15}}{1+x^4} = -f(x)$

Örnek...10 :

$$\int_0^1 x \cdot (x^2+1)^3 dx = ? \quad u = x^2+1$$

$du = 2x dx$

x	0	1
u	1	2

$$\int_1^2 u^3 \frac{du}{2} = \frac{1}{2} \cdot \frac{1}{4} u^4 \Big|_1^2 = \frac{1}{8} (2^4 - 1^4) = \frac{15}{8}$$

Örnek...11 :

Grafiği verilen $y=f(x)$ fonksiyonunun grafiğine

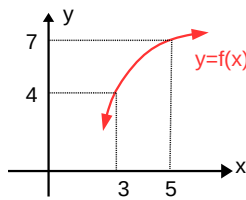
göre $\int_3^5 (f(x) + x \cdot f'(x)) dx = ?$

$\int_3^5 (x f(x))' dx$

$x f(x) \Big|_3^5 = 5 f(5) - 3 f(3)$

$= 35 - 12$

$= 23$



Örnek...12 :

$$\int_2^3 f(x) dx = 5 \text{ ise } \int_2^3 (7-f(x)) dx = ?$$

$$\int_2^3 7 - \int_2^3 f(x) dx$$

$$7x \Big|_2^3 - 5 = 7 \cdot 3 - 7 \cdot 2 - 5 = 2$$

Örnek...13 :

$$\int_0^2 f(4x) dx = 60 \text{ ise } \int_0^8 (1-f(x)) dx = ?$$

$u = 4x$
 $du = 4 dx$

x	0	2
u	0	8

$$\int_0^8 f(u) \cdot \frac{du}{4} = 60$$

$$\int_0^8 f(u) du = 240$$

$$\int_0^8 dx - \int_0^8 f(x) dx$$

$$= x \Big|_0^8 - 240$$

$$= 8 - 240$$

$$= -232$$

Örnek...14 :

$\int_1^{256} \frac{\sqrt[6]{x} - \sqrt[3]{x}}{\sqrt[4]{x}} dx$ integralinde $x=u^{12}$ dönüşümü yaparak tekrar integrali yazınız ($u>0$)

$x = u^{12}$

x	1	256
u	1	$\sqrt[12]{256} = \sqrt[3]{4}$

$$dx = 12 u^{11} du$$

$$\int_1^{\sqrt[3]{4}} \left(\frac{u^2 - u^4}{u^3} \right) \cdot 12 u^{11} du$$

$$12 \int_1^{\sqrt[3]{4}} (u^{10} - u^8) du$$

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Örnek...15 :

$y=f(x)$ fonksiyonu reel sayılarda türevli ve $x=2$ noktasındaki teğeti x eksenine pozitif yönde 45° lik açı yapıyorsa ve $x=3$ ekstremum noktalarından birinin apisi ise

$$\int_2^3 \frac{x \cdot f''(x) - f'(x)}{x^2} dx = ?$$

$$\int_2^3 \left(\frac{f'(x)}{x} \right)' dx = \frac{f'(x)}{x} \Big|_2^3$$

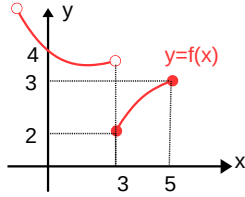
$$= \frac{f'(3)}{3} - \frac{f'(2)}{2} = \frac{0}{3} - \frac{1}{2} = -\frac{1}{2}$$

Örnek...16 :

$y=f(x)$ fonksiyonunun grafiği şekildeki gibidir

$$\int_3^5 f(x) + x \cdot f'(x) dx = ?$$

$$\int_3^5 [x f(x)]' dx = x f(x) \Big|_3^5 = 5f(5) - 3f(3) = 5 \cdot 3 - 3 \cdot 2 = 15 - 6 = 9$$



Örnek...17 :

$\int_0^t (x^2 - 4x - 5) dx$ integralinin alacağı sonuç en küçük değer kaçtır?

$$F(t) = \frac{t^3}{3} - 2t^2 - 5t \quad]_{\min} = ?$$

$$F'(t) = t^2 - 4t - 5 = 0 \quad (t-5)(t+1) = 0 \quad t = 5 \vee t = -1$$

$$F(5) = \frac{125}{3} - 50 - 25 = -\frac{100}{3}$$

ÖZEL TANIMLI FONKSİYONLARIN İNTEGRALI

İntegrandında parçalı fonksiyon veya mutlak değerli fonksiyon içeren integraller integralin alındığı sınırlar içerisinde kritik nokta içeriyorsa göre parçalanarak integralleri alınır.

$f: [a, b] \rightarrow \mathbb{R}$ fonksiyonu $[a, b]$ aralığındaki bulunan sonlu sayıdaki $a_0, a_1, a_2, \dots, a_n$ sayıları için süreksiz ise bu noktalara göre integral parçalanır.

$$\text{Yani } \int_a^b f(x) dx = \int_a^{a_1} f(x) dx + \int_{a_1}^{a_2} f(x) dx + \dots + \int_{a_{n-1}}^{a_n} f(x) dx$$

Bu parçalamayı genelde parçalı fonksiyonda veya mutlak değerli kritik noktasında ihtiyaç duyarsak yaparız

Örnek...18 :

$$f(x) = \begin{cases} (e)^{x^2} & x \geq 0 \\ x & x < 0 \end{cases} \text{ ise } \int_{-4}^{-2} f(x) dx$$

$$\int_{-4}^{-2} x dx = \frac{x^2}{2} \Big|_{-4}^{-2} = \frac{4 - 16}{2} = -6$$

Örnek...19 :

$$f(x) = \begin{cases} x^2 & x < 2 \\ x^3 & 2 < x < 3 \\ x+1 & x \geq 3 \end{cases} \text{ fonksiyonu için } \int_0^5 f(x) dx$$

$$\int_0^2 x^2 dx + \int_2^3 x^3 dx + \int_3^5 (x+1) dx = \frac{x^3}{3} \Big|_0^2 + \frac{x^4}{4} \Big|_2^3 + \frac{x^2}{2} + x \Big|_3^5 = \frac{8}{3} + \frac{65}{4} + 10 = \frac{347}{12}$$

Örnek...20 :

$$f(x) = \begin{cases} 2x+1 & x \geq 0 \\ 3x^2 & x < 0 \end{cases} \text{ ise } \int_{-1}^3 f(x) dx$$

$$\int_{-1}^0 3x^2 dx + \int_0^3 (2x+1) dx = x^3 \Big|_{-1}^0 + x^2 + x \Big|_0^3 = 0 - (-1) + (9+3) - 0 = 1+12 = 13$$

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Örnek...21 :

$$f(x) = \begin{cases} 2^x & x \geq 0 \\ x+1 & x < 0 \end{cases} \text{ ise } \int_{-3}^1 f(x+2) dx$$

$$\begin{aligned} v &= x+2 \\ du &= dx \end{aligned} \quad \begin{array}{c|cc} x & -3 & 1 \\ \hline u & -1 & 3 \end{array}$$

$$\int_{-1}^3 f(u) du = \int_{-1}^0 (x+1) dx + \int_0^3 2^x dx$$

$$\left. \frac{x^2}{2} + x \right|_{-1}^0 - \left. x^2 \right|_0^3 = \frac{1}{2} - 9 = -\frac{17}{2}$$

Örnek...22 :

$$\int_0^4 |x+2| dx = \int_0^4 (x+2) dx = \left. \frac{x^2}{2} + 2x \right|_0^4 = 8 + 8 - 0 = 16$$

Örnek...23 :

$$\int_0^2 x^3 |x-1| dx = \int_0^1 -x^3(x-1) dx + \int_1^2 x^3(x-1) dx$$

$$= \int_0^1 (-x^4 + x^3) dx + \int_1^2 (x^4 - x^3) dx$$

$$= \left. -\frac{x^5}{5} + \frac{x^4}{4} \right|_0^1 + \left. \frac{x^5}{5} - \frac{x^4}{4} \right|_1^2$$

$$= -\frac{1}{5} + \frac{1}{4} + \frac{32}{5} - 4 - \frac{1}{5} + \frac{1}{4}$$

$$= 6 + \frac{1}{2} = \frac{13}{2}$$

Örnek...24 :

$$\int_{-1}^1 (2x-3)|x| dx = \int_{-1}^0 -(2x-3)x dx + \int_0^1 (2x-3)x dx$$

$$= \int_{-1}^0 (-2x^2 + 3x) dx + \int_0^1 (2x^2 - 3x) dx$$

$$= \left. -\frac{2x^3}{3} + \frac{3x^2}{2} \right|_{-1}^0 + \left. \frac{2x^3}{3} - \frac{3x^2}{2} \right|_0^1$$

$$= 0 - \left(\frac{2}{3} + \frac{3}{2} \right) + \left(\frac{2}{3} - \frac{3}{2} \right) - 0 = -\frac{3}{2}$$

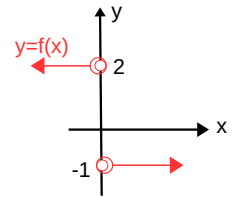
Örnek...25 :

$$\int_0^1 \frac{x^2-4}{|x-2|} dx = \int_0^1 \frac{(x^2-4)}{-(x-2)} dx = - \int_0^1 (x+2) dx$$

$$= - \left(\frac{x^2}{2} + 2x \right) \Big|_0^1 = - \left(\frac{1}{2} + 2 - 0 \right) = -\frac{5}{2}$$

Örnek...26 :

Reel sayılarda sürekli olan f fonksiyonunun türevinin grafiği veriliyor. $f(3) - f(2) = ?$



$$f'(x) = \begin{cases} 2 & x < 0 \\ -1 & x > 0 \end{cases}$$

$$\int_2^3 f'(x) dx = \int_2^3 -1 dx = -x \Big|_2^3 = -(3-2) = -1$$

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DEĞERLENDİRME

$$\int f(x) dx = F(x) + C$$

$$\int g(x) dx = G(x) + C$$

1) $\int_2^3 f(x) dx = 10$ ve $\int_2^3 g(x) dx = 8$ ise

$$\left(\int_2^3 (4f(x) + 3g(x)) dx \right) \cdot \left(\int_2^3 (5f(x) - 4g(x)) dx \right)$$

$$F(3) - F(2) = 10 \quad G(3) - G(2) = 8$$

$$(4[F(3) - F(2)] + 3[G(3) - G(2)]) \cdot (5[F(3) - F(2)] - 4[G(3) - G(2)])$$

$$(40 + 24) \cdot (50 - 32)$$

$$64 \cdot 18 = 1152$$

2) $\int_{-1}^{15} f(x) dx = 10$, ve $\int_{-1}^{15} g(x) dx = 8$ ise

$$\left(\int_{-1}^{15} f(x) + g(x) dx \right) \cdot \left(\int_{-1}^{15} f(x) - g(x) dx \right)$$

$$18 \cdot 2 = 36$$

3) $\int_2^3 f(x) dx = -3$ ve $\int_3^7 f(x) dx = 5$ ise $\int_2^7 f(x) dx = ?$

$$\begin{array}{r} -/ F(3) - F(2) = -3 \\ + F(7) - F(3) = 5 \\ \hline F(7) - F(2) = 8 \end{array}$$

4) $\int_1^2 f(x) dx = 6$ ve $\int_1^2 (5 - 3f(x)) dx = ?$

$$\int_1^2 5 dx - 3 \int_1^2 f(x) dx$$

$$5x \Big|_1^2 - 3 \cdot 6$$

$$5 - 18 = -13$$

5) $\int_1^2 \left(\sum_{n=0}^2 \frac{x^n}{n!} \right) dx = ?$

$$\int_1^2 \left(1 + x + \frac{x^2}{2} \right) dx$$

$$x + \frac{x^2}{2} + \frac{x^3}{6} \Big|_1^2 = \left(2 + 2 + \frac{8}{6} \right) - \left(1 + \frac{1}{2} + \frac{1}{6} \right)$$

$$= \frac{16}{3} - \frac{10}{6} = \frac{22}{6} = \frac{11}{3}$$

6) $\int_{-3}^{-4} \frac{1}{(x+5)^4} dx = ?$

$$u = x+5 \quad du = dx$$

$$\int_2^1 u^{-4} du = \frac{u^{-3}}{-3} \Big|_2^1$$

$$= -\frac{1}{3} \left(1 - \frac{1}{8} \right) = -\frac{7}{24}$$

7) $\int_{-2}^2 (x^{15} + \sin^9 x) dx = ?$

Simetrik aralık
tek fonksiyon

8) $\int_0^{12} f\left(\frac{x}{3}\right) dx = 36$ ise $\int_0^2 (x+3 - 4f(2x)) dx = ?$

$$u = \frac{x}{3} \quad du = \frac{1}{3} dx$$

$$\int_0^4 f(u) \cdot 3 \cdot du = 36$$

$$\int_0^4 f(u) du = 12$$

$$\int_0^2 (x+3) dx - 4 \int_0^4 \frac{f(u) du}{2}$$

$$= \frac{x^2}{2} + 3x \Big|_0^2 - 4 \cdot \frac{12}{2}$$

$$= 8 - 24 = -16$$

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9) $\int_1^4 f(\sqrt{x}) \frac{dx}{\sqrt{x}} = 3$ ise $\int_{\sqrt{2}}^2 5x \cdot f\left(\frac{x^2}{2}\right) dx = ?$

$$\begin{aligned} u = \sqrt{x} \quad du = \frac{1}{2\sqrt{x}} dx \\ \frac{x}{u} \Big|_1^4 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2} \\ \int_1^2 f(u) \cdot \frac{2u du}{u} = 3 \\ \int_1^2 f(u) du = \frac{3}{2} \end{aligned} \quad \left| \quad \begin{aligned} u = \frac{x^2}{2} \quad du = x dx \\ \frac{x}{u} \Big|_{\sqrt{2}}^2 = \frac{2}{1} - \frac{\sqrt{2}}{2} = \frac{4-\sqrt{2}}{2} \\ 5 \int_{\sqrt{2}}^2 f(u) du = 5 \cdot \frac{3}{2} = \frac{15}{2} \end{aligned} \right.$$

10) $\int_1^{64} \left(\frac{\sqrt{x}-x}{\sqrt[3]{x}} \right) dx$ integralinde $x=u^6$ dönüşümü yaparak tekrar integrali yazınız

$$\begin{aligned} x = u^6 \quad dx = 6u^5 du \\ \frac{x}{u} \Big|_1^{64} = \frac{64}{2} - \frac{1}{2} = \frac{63}{2} \\ \int_1^2 \left(\frac{u^3 - u^6}{u^2} \right) \cdot 6u^5 du = 6 \int_1^2 (u^6 - u^9) du \\ = 6 \left(\frac{u^7}{7} - \frac{u^{10}}{10} \right) \Big|_1^2 = \frac{6}{7} \left(128 - \frac{1024}{10} \right) = \frac{6}{7} \left(128 - 102.4 \right) = \frac{6}{7} \cdot 25.6 = \frac{153.6}{7} \end{aligned}$$

11) $y=f(x)$ fonksiyonu reel sayılarda türevli ve $x=1$ noktasındaki teğeti x eksenine paralel ve $x=5$ deki teğeti $y-3x+2=0$ doğrusuna dikse =?

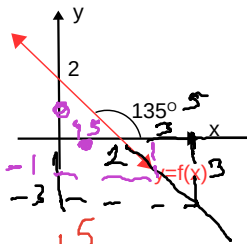
$$\begin{aligned} \int_1^5 \frac{f''(x)}{x} dx + \int_5^1 \frac{f'(x)}{x^2} dx \\ \int_1^5 \left(\frac{x \cdot f''(x) - f'(x)}{x^2} \right) dx \\ \int_1^5 \left[\frac{f'(x)}{x} \right]' dx = \left. \frac{f'(x)}{x} \right|_1^5 = \frac{f'(5)}{5} - \frac{f'(1)}{1} \\ = \frac{-1/3}{5} - 0 = -\frac{1}{15} \end{aligned}$$

12) $y=f(x)$ fonksiyonu n grafiği

şekildeki gibidir

$\int_3^5 f(x) + x \cdot f'(x) dx$ integralinin değerini bulunuz

$$\begin{aligned} \int_3^5 [x f(x)]' dx &= x f(x) \Big|_3^5 \\ &= 5 f(5) - 3 f(3) \\ &= 5 \cdot (-3) - 3 \cdot (-1) \\ &= -15 + 3 = -12 \end{aligned}$$



13) $f(x) = \begin{cases} 1+x & x < 1 \\ x & 1 \leq x \leq 3 \\ x-1 & x > 3 \end{cases}$ fonksiyonu için $\int_0^4 f(x) dx$

$$\begin{aligned} \int_0^1 (1+x) dx + \int_1^3 x dx + \int_3^4 (x-1) dx \\ x + \frac{x^2}{2} \Big|_0^1 + \frac{x^2}{2} \Big|_1^3 + \frac{x^2}{2} - x \Big|_3^4 \\ 1 + \frac{1}{2} + \frac{9}{2} - \frac{1}{2} + (8-4) - \left(\frac{9}{2} - 3 \right) = \frac{8}{2} \end{aligned}$$

14) $f(x) = \begin{cases} 1+2x & x \geq 4 \\ 3x & x < 4 \end{cases}$ fonksiyonu için $\int_{-1}^3 f(x+3) dx$

$$\begin{aligned} u = x+3 \quad du = dx \\ \frac{x}{u} \Big|_{-1}^3 = \frac{3}{6} - \frac{-1}{2} = \frac{1}{2} + \frac{1}{2} = 1 \\ \int_{-1}^3 f(u) du = \int_{-1}^1 3u du + \int_1^3 (1+2u) du \\ \frac{3u^2}{2} \Big|_{-1}^1 + \left(u + u^2 \right) \Big|_1^3 \\ \frac{3}{2} (1 - 1) + (3 + 9) - (1 + 1) = \frac{40}{2} = 20 \end{aligned}$$

15) $\int_1^4 |x-2| dx$

$$\begin{aligned} \int_1^2 (-x+2) dx + \int_2^4 (x-2) dx \\ -\frac{x^2}{2} + 2x \Big|_1^2 + \frac{x^2}{2} - 2x \Big|_2^4 \\ (-2+4) - \left(-\frac{1}{2} + 2 \right) + \left((8-8) - (2-4) \right) \\ \frac{1}{2} + 2 = \frac{5}{2} \end{aligned}$$

16) $\int_{-1}^3 x \cdot |x-2| dx$

$$\begin{aligned} \int_{-1}^2 -x(x-2) dx + \int_2^3 x(x-2) dx \\ \int_{-1}^2 (-x^2 + 2x) dx + \int_2^3 (x^2 - 2x) dx \\ -\frac{x^3}{3} + x^2 \Big|_{-1}^2 + \frac{x^3}{3} - x^2 \Big|_2^3 \\ \left(-\frac{8}{3} + 4 \right) - \left(-\frac{1}{3} + 1 \right) + \left(\frac{27}{3} - 9 \right) - \left(\frac{8}{3} - 4 \right) \\ 0 + \frac{4}{3} = \frac{4}{3} \end{aligned}$$